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MOMENT METHOD FOR SOLUTION OF THE
SCHWARZSCHILD-MILNE
INTEGRAL EQUATION

JEAN I. F. KING AND E. T. FLORANCE

SCIENTIFIC REPORT NO. 1
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ABSTRACT

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The underlying basis of the extreme accuracy of the double-Gauss quadrature formula devised in the method of discrete ordinates is uncovered in an alternative solution of the transfer equation. The Schwarzschild-Milne integral equation is solved by approximating the exponential integral kernel with a finite sum of exponential functions. A moment method is used to provide the best fit to the kernel. The constants that result are identical to those following from the choice of a double-Gauss quadrature formula in the discrete ordinate method. The integral equation formalism is then applied to the non-gray atmosphere problem.

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I. INTRODUCTION

The method of discrete ordinates developed by Wick (1943) and Chandrasekhar (1950) is a powerful technique for the solution of transfer equations. A critical factor is the choice of a proper quadrature formula to replace the integration of radiation intensity over direction. Sykes (1951) obtained results of extreme accuracy by splitting the interval and fitting the Gaussian formula separately over the upward and downward directions.

The physical basis for the success of the double-Gauss method is laid bare by an alternative solution of the equilibrium transfer equation which does not involve the intensity. The Schwarzschild-Milne integral equation is approximately solved by expanding the kernel transmittance in a summation of exponential functions. The characteristic equation that results is formally identical with that of the method of discrete ordinates. The specification of a "best fit" of the kernel and its approximate representation by equating moments, leads to a set of equations which reduce to the Legendre polynomials of the double-Gauss method. Thus the ad hoc choice of the double-Gauss formula is justified as providing the optimum fit of the exponential integral kernel by a finite sum of exponential functions.

II. MOMENT METHOD SOLUTION OF THE SCHWARZSCHILD-MILNE EQUATION

The transfer equation specifies a relation between the radiation intensity and Planck source function which, for a plane-parallel, non-scattering, gray atmosphere may be written as

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - B(\tau) \quad (1)$$

The imposition of the equilibrium constraint

$$B(\tau) = \frac{1}{2} \int_{-1}^1 I(\tau, \mu) d\mu \quad (2)$$

permits the elimination of either of these dependent variables. Thus, by substituting the source function (2) into Equation (1) we obtain the equilibrium integrodifferential equation of transfer

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^1 I(\tau, \mu') d\mu' \quad (3)$$

The method of discrete ordinates solves the problem approximately by converting the integrodifferential equation into a system of $2n$ linear differential equations. This is done by replacing the integration over direction with a suitably chosen quadrature formula

$$\int_{-1}^1 I(\tau, \mu) d\mu \approx \sum_i a_i I(\tau, \mu_i) \quad (i = 1, \dots, \pm n) \quad (4)$$

Chandrasekhar's (1950) use of a Gaussian quadrature formula was criticized by Kourganoff (1952) who preferred the Newton-Cotes method. Sykes (1951), meanwhile, obtained results of extreme accuracy by splitting the interval and fitting the Gaussian formula separately over the ranges $(-1,0)$ and $(0,1)$. We now demonstrate the physical basis underlying Sykes' choice of a double-Gauss method, showing how it represents the optimum choice of a polynomial quadrature formula.

Returning to Equations (1) and (2), we can use the equilibrium constraint alternatively to eliminate the intensity from the integral form of the transfer equation yielding [see Chandrasekhar (1950)]

$$B(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} B(t) E_1(|t-\tau|) dt, \quad (5)$$

the Schwarzschild-Milne integral equation. The direct solution of this equation is difficult. The form of the kernel

$$E_1(\tau) \equiv \int_0^1 e^{-\tau/\mu} \frac{d\mu}{\mu} \approx \sum_{i=1}^n \frac{a_i}{\mu_i} e^{-\tau/\mu_i}, \quad (6)$$

suggests an approximate expansion into a summation of more tractable exponential functions.

Equation (5) becomes, with this kernel approximation,

$$B(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} B(t) \sum_{i=1}^n \frac{a_i}{\mu_i} e^{-|t-\tau|/\mu_i} dt. \quad (7)$$

The application of the Laplace transform using the Faltung theorem [Sneddon (1951)] leads directly to

$$B(k) = \frac{1}{2} B(k) \sum_{i=1}^n \left(\frac{a_i}{1+\mu_i k} + \frac{a_i}{1-\mu_i k} \right), \quad (8)$$

where

$$B(k) \equiv \int_0^{\infty} B(\tau) e^{-k\tau} d\tau. \quad (9)$$

The requirement that $B(k) \neq 0$ then yields as the characteristic equation

$$1 - \sum_{i=1}^n \frac{a_i}{1-\mu_i^2 k^2} = 0, \quad (10)$$

whose $2n - 1$ solutions consist of a double root at the origin $k^2 = 0$ and paired roots at $k = \pm k_{\alpha}$.

We obtain the general solution for the equilibrium source function by performing an inversion of Equation (8) with the poles along the real axis given by the roots of the characteristic equation. Thus

$$B(\tau) = b \left\{ \tau + Q + \sum_{\alpha=1}^{n-1} \left[L_{\alpha} e^{-k_{\alpha} \tau} + L_{-\alpha} e^{k_{\alpha} \tau} \right] \right\}. \quad (11)$$

The constants b , Q , L_α , and $L_{-\alpha}$ can be determined by boundary conditions in either of two ways. For a semi-infinite atmosphere the Wiener-Hopf technique can be used to express the constants directly as residues of the H-functions (King 1955), yielding

$$b = 3F/4 \quad ,$$

$$Q = \sum_{i=1}^n \mu_i - \sum_{\alpha=1}^{n-1} \frac{1}{k_\alpha} \quad , \quad (12)$$

$$L_\alpha = \lim_{\mu \rightarrow 1/k_\alpha} \frac{(1 - k_\alpha \mu)}{\sqrt{3}} H(-\mu) \quad ,$$

$$L_{-\alpha} = 0 \quad ,$$

where the H-function is given in this approximation as

$$H(\mu) = \frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_{i=1}^n (\mu + \mu_i)}{\prod_{\alpha=1}^{n-1} (1 + k_\alpha \mu)} \quad . \quad (13)$$

Alternatively one can determine the constants by the requirement that $B(\tau) = 0$ for $\tau < 0$. This constraint leads to a set of linear simultaneous equations to determine the n constants Q , L_α . Upon using the method of elimination of constants we are led then to relations (12).

The characteristic equation (10) and constant relations (12) derived from the Schwarzschild-Milne equation are formally identical to those obtained by Chandrasekhar (1950) using the method of discrete ordinates. This is not surprising since the two approaches are transformations of the same problem.

The result is more than an elegant identity. Firstly, we have derived the equilibrium source function $B(\tau)$ directly without recourse to any auxiliary function such as the radiation intensity. More importantly, however, we have in the kernel approximation, Equation (6), an algorithm for the specification of the best quadrature formula.

We return to the kernel approximation and determine the weights a_i and directions μ_i by equating moments of the kernel with its series approximation. Thus

$$\int_0^{\infty} E_1(\tau) \tau^{\ell} d\tau = \sum_{i=1}^n \frac{a_i}{\mu_i} \int_0^{\infty} e^{-\tau/\mu_i} \tau^{\ell} d\tau, \quad (14)$$

yielding the following system of nonlinear equations to determine the $2n$ constants a_i, \dots, a_n and μ_1, \dots, μ_n

$$\sum_{i=1}^n a_i \mu_i^{\ell} = E_{\ell+2}(0) = \frac{1}{\ell+1} \quad (\ell=0, 1, \dots, 2n-1) \quad (15)$$

III. SOLUTION OF MOMENT EQUATIONS

A method for solving moment equations of the type

$$\sum_{i=1}^n a_i \mu_i^\ell = b_\ell \quad (\ell = 0, \dots, 2n-1) \quad , \quad (16)$$

has been given by Chandrasekhar (1950). He shows that if coefficients c_j ($j = 0, \dots, n-1$) are defined by the linear equations

$$b_{n+\ell} + \sum_{j=0}^{n-1} c_j b_{j+\ell} = 0 \quad (\ell = 0, \dots, n-1) \quad , \quad (17)$$

then μ_i is one of the n roots of the polynomial

$$F(x) \equiv x^n + \sum_{j=0}^{n-1} c_j x^j \quad . \quad (18)$$

The coefficients c_j can be eliminated from (17) and (18), with the result that $F(x)$ is a multiple of the polynomial

$$\Phi(x) \equiv \det \left[\tilde{b}_{j\ell}(x) \right] \quad , \quad (19)$$

where $\tilde{b}(x)$ is the $(n+1) \times (n+1)$ matrix

$$\tilde{b}_{j\ell} = b_{j+\ell} \quad (j = 0, \dots, n-1), \quad b_{n\ell} = x^\ell \quad (\ell = 0, \dots, n). \quad (20)$$

The determinantal equation $\Phi(x) = 0$ in which we have from (15)

$$b_\ell = \frac{1}{\ell+1} = \int_0^1 dx x^\ell \quad (21)$$

can be simplified (Muir 1960) to the form

$$\frac{d^n}{dx^n} \left[x^n (1-x)^n \right] = 0 \quad . \quad (22)$$

The substitution

$$x = \frac{1}{2}(1 + \bar{x}) \quad (23)$$

immediately reduces (22) to the equation

$$P_n(\bar{x}) = 0 \quad ,$$

where P_n is the Legendre polynomial of order n . Thus, we have arrived at the same result as the Sykes double-Gauss method:

$$\mu_i = \frac{1}{2}(1 + \bar{\mu}_i) \quad , \quad (24)$$

where $\bar{\mu}_i$ is one of the n roots of the Legendre polynomial P_n .

Since the transformation (23) maps the interval $(-1, 1)$ onto $(0, 1)$, Sykes (1951) noted that the double-Gauss formula is merely the Gauss formula applied to the transformed interval. We will now demonstrate that the linear mapping (23) correctly transforms the moment equations (16).

If we insert (23) into the definition

$$\bar{b}_\ell = \int_{-1}^1 d\bar{x} \quad \bar{x}^\ell$$

and use the moment equations (16) with (24) we obtain the transformed equations

$$\sum_{i=1}^n \bar{a}_i \bar{\mu}_i^{\ell} = \bar{b}_{\ell} \quad , \quad (25)$$

where $\bar{a}_i = 2a_i$. In other words, a solution to the equations (16) for the interval (0, 1) is a linear function of the solution to (25) on the interval (-1, 1).

The solution to (25) with

$$\begin{aligned} \bar{b}_{\ell} &= 0 (\ell \text{ odd}) \quad , \\ \bar{b}_{\ell} &= \frac{2}{\ell+1} (\ell \text{ even}) \quad , \end{aligned}$$

is the familiar Gauss formula (Chandrasekhar 1950). Hence, the weights a_i are given by

$$a_i = \frac{1}{2} \frac{1}{P'_n(\mu_i)} \int_{-1}^1 d\bar{\mu} \frac{P_n(\bar{\mu})}{\bar{\mu} - \bar{\mu}_i} \quad . \quad (26)$$

Writing (26) in the form

$$a_i = \left[\frac{dP_n}{d\mu_i} (2\mu_i - 1) \right]^{-1} \int_0^1 d\mu \frac{P_n(2\mu-1)}{\mu - \mu_i}$$

reveals the standard prescription of a_i as a quadrature weight.

The characteristic Equation (10) has been solved up to the fourth approximation. The roots μ_i and k_α along with the derived constants Q and L_α from (12) are given for reference in Table 1. As an indication of the superior accuracy of the double-Gauss method, in Table 2 we compare the fourth approximation for $q(\tau)$ and $H(\mu)$ with the exact result from Kourganoff (1952; Tables 33, 34, p. 138).

TABLE 1

CHARACTERISTIC ROOTS AND CONSTANTS OF INTEGRATION

Approximation	Direction Cosine	Characteristic Roots	Q-constant	L-constants
First	$\mu_1 = .50000000$	+ .500000
Second.	$\mu_1 = .21132487$ $\mu_2 = .78867513$	$k_1 = 3.46410162$	+ .711325	$L_1 = -.133975$
Third	$\mu_1 = .11270167$ $\mu_2 = .50000000$ $\mu_3 = .88729834$	$k_1 = 1.52028083$ $k_2 = 7.59531080$	+ .710567	$L_1 = -.057038$ $L_2 = -.076179$
Fourth.	$\mu_1 = .06943184$ $\mu_2 = .33000948$ $\mu_3 = .66999052$ $\mu_4 = .93056816$	$k_1 = 13.12234030$ $k_2 = 1.22721388$ $k_3 = 2.50960776$	+ .710472	$L_1 = -.048082$ $L_2 = -.024173$ $L_3 = -.060866$

TABLE 2

COMPARISON OF FOURTH APPROXIMATION AND EXACT SOLUTIONS

τ	$q_{ap}(\tau)$	$q_{ex}(\tau)$	μ	$H_{ap}(\mu)$	$H_{ex}(\mu)$
0.00	.57735	.577351	0.0	1.00000	1.0000000
0.01	.58507	.588236	0.1	1.24619	1.2473502
0.02	.59202	.595391	0.2	1.45020	1.4503515
0.04	.60396	.606287	0.3	1.64257	1.6425221
0.06	.61378	.614789	0.4	1.82937	1.8292757
0.08	.62194	.621854	0.5	2.01288	2.0127788
0.10	.62879	.627919	0.6	2.19423	2.1941330
0.20	.65123	.649550	0.7	2.37407	2.3739750
0.40	.67312	.673090	0.8	2.55279	2.5527044
0.60	.68537	.685801	0.9	2.73067	2.7305876
0.80	.69324	.693535	1.0	2.90789	2.9078105
1.00	.69844	.698540			
2.00	.70799	.707916			
3.00	.70983	.709806			
∞	.71047	.710447			

IV. EXTENSION TO NON-GRAY ATMOSPHERES

The Schwarzschild-Milne integral equation approach shows to best advantage in an extension to the non-gray atmosphere problem. The flux transmittance for a narrow band of width $\Delta\nu$ consisting of many lines, yet small enough so that the Planck function is sensibly constant, is defined as

$$\begin{aligned} 2\mathcal{E}_3(\tau) &= \frac{2}{\Delta\nu} \int_{\Delta\nu} \int_0^1 e^{-\kappa_\nu u/\mu} \mu d\mu d\nu \\ &= 2 \int_0^1 \int_0^1 e^{-\tau/\mu\lambda} \mu d\mu d(\nu/\Delta\nu) , \end{aligned} \quad (27)$$

where $\lambda = \kappa/\kappa_\nu$ is the ratio of the arithmetic mean and monochromatic absorption coefficients.

By analogy with the gray case we can define a hierarchy of non-gray transmittances by

$$\mathcal{E}_n(\tau) = \int_0^1 \int_0^1 e^{-\tau/\mu\lambda} (\mu\lambda)^{n-2} d\mu \frac{d(\nu/\Delta\nu)}{\lambda} , \quad (28)$$

obeying the relation

$$\frac{d\mathcal{E}_n(\tau)}{d\tau} = -\mathcal{E}_{n-1}(\tau) \quad (n = 2, 3, \dots) . \quad (29)$$

The non-gray Schwarzschild-Milne equation becomes

$$B(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} B(t) \mathcal{E}_1(|t-\tau|) dt \quad . \quad (30)$$

As we continue to exploit the analogy, the form of the equation suggests that we expand the kernel in a summation of exponential functions

$$\mathcal{E}_1(\tau) \approx \sum_{i=1}^n \frac{a_i}{\mu_i} e^{-\tau/\mu_i} \quad , \quad (31)$$

where the a_i and μ_i are now "generalized" weights and roots.

Equating moments as before, these constants are determined by the system of $2n$ nonlinear simultaneous equations

$$\sum_{i=1}^n a_i \mu_i^\ell = \mathcal{E}_{\ell+2}(0) \equiv \beta_\ell \quad (\ell = 0, 1, \dots, 2n-1). \quad (32)$$

Thus the non-gray problem is reduced simply to the determination of new constants specified by different moments. Using relation (27) we may tabulate these as

$$\begin{aligned} \mathcal{E}_2(0) &= 1, \quad \mathcal{E}_3(0) = \frac{1}{2}, \quad \mathcal{E}_4(0) = \frac{1}{3}, \quad \mathcal{E}_5(0) = \frac{1}{4}, \dots, \\ \mathcal{E}_n(0) &= \frac{1}{(n-1)} \quad . \end{aligned} \quad (33)$$

As an example consider an Elsasser band model consisting of an equally-spaced array of identical Lorentz lines whose monochromatic absorption coefficient is

$$\kappa_{\nu} = \frac{S}{d} \frac{\sinh(2\pi\alpha/d)}{\cosh(2\pi\alpha/d) - \cos(2\pi\nu/d)} , \quad (34)$$

where S , α , and d are the line intensity, halfwidth, and spacing. By integrating from line center to center we obtain the relations

$$\kappa = \frac{S}{d}, \quad \bar{\lambda} = \frac{\kappa}{\kappa_h} = \coth \beta, \quad \bar{\lambda}^2 = \frac{\cosh^2 \beta + \frac{1}{2}}{\sinh^2 \beta} , \quad (35)$$

where $\bar{\lambda}$, the non-grayness parameter, is the ratio of the arithmetic and harmonic means of the absorption coefficient and $\beta = 2\pi\alpha/d$ is the ratio of line half-width and spacing. Any deviation from grayness leads to a $\bar{\lambda}$ in excess of unity.

Although the μ_i in Equation (32) are no longer roots of the Legendre polynomials as in the gray case, it is not difficult to solve the equation set explicitly in the $n = 2$ approximation. Doing this we find μ_1, μ_2 as roots of the quadratic equation

$$(\beta_0\beta_2 - \beta_1^2) \mu^2 - (\beta_0\beta_3 - \beta_1\beta_2) \mu + (\beta_1\beta_3 - \beta_2^2) = 0, \quad (36)$$

with the root of the characteristic equation given by

$$k_1 = \sqrt{\frac{\beta_2}{\beta_0}} \frac{\beta_0\beta_2 - \beta_1^2}{\beta_1\beta_3 - \beta_2^2} . \quad (37)$$

The most interesting result, identical in all orders of approximation and hence exact, is the Hopf-Bronstein relation generalized now for a non-gray atmosphere. Thus

$$q(0) \equiv Q + L_1 = \mu_1 \mu_2^k = \sqrt{\frac{\beta_2}{\beta_o}} = \sqrt{\frac{\bar{\lambda}}{3}} . \quad (38)$$

The remaining constants in this approximation are determined as

$$Q = \mu_1 + \mu_2 - \frac{1}{k_1} = \frac{\beta_o \beta_3 - \beta_1 \beta_2 - (\beta_o / \beta_2)^{\frac{1}{2}} (\beta_1 \beta_3 - \beta_2^2)}{\beta_o \beta_2 - \beta_1^2}, \quad (39)$$

$$L_1 = \sqrt{\frac{\beta_2}{\beta_o}} - Q .$$

V. CONCLUSIONS

The double-Gauss quadrature formula used by Sykes (1951) has a sound physical basis in providing the optimum fit of the kernel in the Schwarzschild-Milne integral equation by an exponential function series. The power of the alternative integral equation formulation is demonstrated by the ease in the extension to treat non-gray atmospheres.

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